Tangency quantum cohomology

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Abstract

Let X be a smooth projective variety. Using modified psi classes on the stack of genus zero stable maps to X, a new associative quantum product is constructed on the cohomology space of X. When X is a homogeneous variety, this structure encodes the characteristic numbers of rational curves in X, and specialises to the usual quantum product upon resetting the parameters corresponding to the modified psi classes. For $X = \mathbb{P}^2$, the product is equivalent to that of the contact cohomology of Ernström-Kennedy.

Introduction

Let X be a smooth projective variety over the complex numbers. The Gromov-Witten invariants of X are constructed by pulling back cohomology classes to the stack of stable maps to X and integrating over the virtual fundamental class (cf. Behrend-Manin [2], Behrend [1], Li-Tian [11]). Let Φ denote the generating function for the genus zero Gromov-Witten invariants. The quantum product is defined on the cohomology space of X by taking the third derivatives of Φ as structure constants,

$$T_i * T_j := \sum_{e,f} \Phi_{ije}\, g^{ef}\, T_f,$$

cf. Kontsevich-Manin [10]. The associativity of this product is equivalent to the fact that Φ satisfies the WDVV equations

$$\sum_{e,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl} \quad = \quad \pm \sum_{e,f} \Phi_{jke}\, g^{ef}\, \Phi_{fil}.$$

A feature of this structure is that when X is a homogeneous variety, the Gromov-Witten invariants are solutions to enumerative problems of rational curves in X subject to incidence conditions, and then the WDVV equations provide relations among the numbers.

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It is straightforward to generalise the construction by including psi classes in the integrals, to get a larger quantum product: instead of using just the primary Gromov-Witten potential as above, one uses the whole gravitational Gromov-Witten potential. This product is associative too, due to the (generalised) WDVV equations (Witten [14]). This larger product has not brought about as much interest as the first one, for two reasons: the gravitational descendants are not directly interpretable in enumerative geometry, and the equations corresponding to associativity are "perpendicular to the descent" in the sense that each equation relates only invariants with the same psi classes as factors.

The present work introduces a new extension of the quantum product, the *tangency quantum product*, and proves its associativity. When X is a homogeneous variety, this structure encodes all the characteristic numbers of rational curves in X.

The construction is based on a different potential Γ, called the *tangency quantum potential*, which incorporates one level of *modified psi classes*. Modified psi classes are boundary corrections of the tautological psi classes motivated from enumerative geometry: tangency conditions are easily expressible in terms of modified psi classes. They were first discovered and studied by R. Pandharipande (unpublished [13]) who used them to establish a topological recursion for the characteristic numbers of rational curves in $\mathbb{P}^2$, and later, independently, by myself [8]. (This work is subsumed in Graber-Kock-Pandharipande [7]).

In contrast to the tautological psi classes, the modified ones restrict to the boundary in a non-trivial way, giving rise to certain *diagonal classes*. Therefore the splitting lemma (1.5), which is the engine in the proof of the associativity, is complicated by correction terms. This mess is handled by the introduction of a new "metric" γ_{ef} on the cohomology space of X. It is a deformation of the Poincaré metric g_{ef} in much the same way as the tangency quantum potential is a deformation of the Gromov-Witten potential. Amazingly it encodes all the combinatorics of the diagonal corrections in the splitting lemma, leading to certain "deformed" WDVV equations. These in turn are equivalent to associativity of the *tangency quantum product*, defined as

$$T_i * T_j := T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\, \gamma^{ef}\, T_f.$$

(This deformed metric also clarifies the topological recursion, as explained in [7].)

The last section is devoted to a technical detail: since modified psi classes are not defined in degree zero, the same goes for the tangency potential. The missing degree zero part is now constructed separately to account for the classical part of the new product. This construction shows that the new product is in fact "integrable" in the sense that its structure constants (with respect to γ) are the third derivatives of a single function. With this last piece of data, the results of the paper can be summarised in saying that the new structure is that of a formal Frobenius (super)manifold (over the power series ring in the new variables).

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1 First enumerative descendants

1.1 Set-up. Throughout we work over the field of complex numbers. Let X denote a smooth projective variety, and let $T_0, \ldots, T_r$ denote the elements of a homogeneous basis of the cohomology space $H^*(X, \mathbb{Q})$. As in Manin's book [12] we consider $H^*(X, \mathbb{Q})$ as a linear supermanifold.

Let $\overline{M}_{0,S}(X, \beta)$ denote the moduli stack of Kontsevich stable maps of genus zero whose direct image in X is of class $\beta \in H_2^+(X, \mathbb{Z})$, and whose marking set is $S = \{p_1, \ldots, p_n\}$. For each mark p_i, let $\nu_i : \overline{M}_{0,S}(X, \beta) \to X$ denote the evaluation morphism that sends the class of a map μ to $\mu(p_i)$; pull-backs along evaluation morphisms of cohomology classes of X are referred to as *evaluation classes*. The reader is referred to Fulton-Pandharipande [5] for definitions and basic properties of stable maps, Gromov-Witten invariants and quantum cohomology.

1.2 Modified psi classes (cf. [7]). Let $\pi_0 : \overline{M}_{0,S\cup\{p_0\}}(X, \beta) \to \overline{M}_{0,S}(X, \beta)$ denote the forgetful morphism that forgets the extra mark p_0. Together with the sections σ_i corresponding to each of the marks in S, it constitutes the universal family. For each mark there is a psi class defined as $\boldsymbol{\psi}_i := c_1(\sigma_i^* \omega_{\pi_0})$, where ω_{π_0} is the relative dualising sheaf of π_0. When $\beta \neq 0$, the *modified psi class* is $\overline{\boldsymbol{\psi}}_i := \hat{\pi}_i^* \boldsymbol{\psi}_i$, where $\hat{\pi}_i : \overline{M}_{0,S}(X, \beta) \to \overline{M}_{0,\{p_i\}}(X, \beta)$ is the forgetful morphism that forgets all marks but p_i.

The importance of modified psi classes comes from enumerative geometry: for example, if X is a homogeneous variety and $Z \subset X$ is a very ample hypersurface of degree $\mathbf{z} \in H^2(X, \mathbb{Q})$, then the cycle of maps tangent to Z at p_i is of class $\nu_i^*(\mathbf{z}) \cup (\overline{\boldsymbol{\psi}}_i + \nu_i^*(\mathbf{z}))$. It then follows from a transversality argument that the characteristic numbers of rational curves are top products of evaluation classes and modified psi classes. For this reason, these top products (integrated over the virtual fundamental class of $\overline{M}_{0,S}(X, \beta)$),

$$\langle\, \overline{\tau}_{m_1}(\mathbf{z}_1) \cdots \overline{\tau}_{m_n}(\mathbf{z}_n)\, \rangle_\beta := \int \overline{\boldsymbol{\psi}}_1^{m_1} \cup \nu_1^*(\mathbf{z}_1) \cup \cdots \cup \overline{\boldsymbol{\psi}}_n^{m_n} \cup \nu_n^*(\mathbf{z}_n) \ \cap [\overline{M}_{0,S}(X, \beta)]^{\mathrm{virt}}$$

are called *enumerative descendants*.

The enumerative descendants that appear in the characteristic number problem are only the *first* enumerative descendants, i.e. those made up of factors of type $\overline{\tau}_0(\mathbf{x})$ and $\overline{\tau}_1(\mathbf{y})$, for $\mathbf{x}, \mathbf{y} \in H^*(X, \mathbb{Q})$. For simplicity, we will work in the fixed basis $T_0, \ldots, T_r$ and employ the following notation for the corresponding integrals:

$$\langle\, \overline{\boldsymbol{\tau}}_0^{\mathbf{a}}\ \overline{\boldsymbol{\tau}}_1^{\mathbf{b}}\, \rangle_\beta := \langle\ \prod_{k=0}^{r} (\overline{\tau}_0(T_k))^{a_k} \prod_{\ell=0}^{r} (\overline{\tau}_1(T_\ell))^{b_\ell}\ \rangle_\beta, \tag{1}$$

where $\mathbf{a} = (a_0, \ldots, a_r)$ and $\mathbf{b} = (b_0, \ldots, b_r)$ are vectors of non-negative integers. The integral is zero unless $\sum_k \frac{1}{2} \deg(T_k)(a_k + b_k + 1) = \mathrm{vdim}_{\mathbb{C}}\, \overline{M}_{0,S}(X, \beta)$, where S is of cardinality $n = \sum(a_k + b_k)$.

1.3 Diagonal classes (cf. [7]). The *diagonal class* $\boldsymbol{\delta}_{ij} \in H^2(\overline{M}_{0,S}(X, \beta), \mathbb{Q})$ is defined as the sum of all boundary divisors having p_i and p_j together on a contracting twig. It can also be described as the pull-back from $\overline{M}_{0,\{p_i,p_j\}}(X, \beta)$ of the divisor D_{ij} having both

marks on a contracting twig; from this description it is clear that diagonal classes are invariant under pull-back along forgetful morphisms.

Let $\pi_0 : \overline{M}_{0,S\cup\{p_0\}}(X,\beta) \to \overline{M}_{0,S}(X,\beta)$ denote the morphism that forgets p_0. Then

$$\pi_{0*}\boldsymbol{\delta}_{0i} = 1, \tag{2}$$

(the virtual fundamental class of $\overline{M}_{0,S}(X,\beta)$).

The diagonal classes owe their name to the following properties.

$$\begin{aligned} \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ik} &= \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{jk} \\ -\boldsymbol{\delta}_{ij}^2 = \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_i &= \boldsymbol{\delta}_{ij}\overline{\boldsymbol{\psi}}_j \\ \boldsymbol{\delta}_{ij}\nu_i^*(\mathbf{z}) &= \boldsymbol{\delta}_{ij}\nu_j^*(\mathbf{z}), \end{aligned} \tag{3}$$

for $\mathbf{z} \in H^*(X,\mathbb{Q})$. The swapping property also holds when $\boldsymbol{\delta}_{ij}$ is replaced with D_{ij}.

1.4 Key formula. Let $D = D(S',\beta' \mid S'',\beta'')$ denote the boundary divisor which is the image of the morphism

$$\rho_D : \overline{M}_{0,S'\cup\{x'\}}(X,\beta') \times_X \overline{M}_{0,S''\cup\{x''\}}(X,\beta'') \longrightarrow \overline{M}_{0,S}(X,\beta)$$

consisting in gluing together the two maps at x' and x''. We will assume that S' and S'' are non-empty and that β' and β'' are non-zero. Then ρ_D is birational onto D. For short, let $\overline{M}'$ and $\overline{M}''$ denote the factors of the fibred product above, and let

$$\jmath_D : \overline{M}' \times_X \overline{M}'' \longrightarrow \overline{M}' \times \overline{M}''$$

denote the inclusion in the cartesian product.

Now there is the following formula for restricting a modified psi class to the boundary (assuming $p_i \in S'$).

$$\rho_D^*\overline{\boldsymbol{\psi}}_i = \jmath_D^*\big(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ix'}\big). \tag{4}$$

1.5 Lemma. Splitting lemma for first enumerative descendants. *Let $D = D(S',\beta'|S'',\beta'')$ be a boundary divisor with β' and β'' non-zero. Let $\mathbf{x}$ and $\mathbf{y}$ be generic even elements of $H^*(X,\mathbb{Q})$, and let there be given a product $\overline{\tau}_0(\mathbf{x})^a\,\overline{\tau}_1(\mathbf{y})^b$ such that the marks corresponding to $a' \le a$ and $b' \le b$ belong to S', and the remaining marks, corresponding to $a'' \le a$ and $b'' \le b$, belong to S''. Then*

$$\begin{aligned} \langle\, D \cdot \overline{\tau}_0(\mathbf{x})^a\,\overline{\tau}_1(\mathbf{y})^b \,\rangle_\beta = \sum_{p,q}\sum_{s',s''} \tbinom{b'}{s'}\tbinom{b''}{s''} \,\langle\, \overline{\tau}_0(\mathbf{x})^{a'}\,\overline{\tau}_1(\mathbf{y})^{b'-s'}\,\overline{\tau}_0(\mathbf{y}^{s'}\cup T_p)\,\rangle_{\beta'} \\ g^{pq}\;\langle\, \overline{\tau}_0(T_q\cup \mathbf{y}^{s''})\,\overline{\tau}_0(\mathbf{x})^{a''}\,\overline{\tau}_1(\mathbf{y})^{b''-s''}\,\rangle_{\beta''}. \end{aligned}$$

The outer sum is over the splitting indices p and q running from 0 to r, and the inner sum is over all s' and s'': the symbol $\overline{\tau}_1(\mathbf{y})^{b'-s'}$ makes sense only for $s' \le b'$, but since there is a binomial coefficient $\binom{b'}{s'}$ in front of it, which is zero unless $s' \le b'$ we will allow any s'. Ditto for s''.

Proof. There are four ingredients in the proof of this formula: the first is of course the splitting axiom of Gromov-Witten theory (cf. Kontsevich-Manin [10]), which accounts for the overall shape of the formula. Second, formula (4) tells how each factor $\overline{\tau}_1(\mathbf{y})$ (say realised at mark p_i as $\overline{\boldsymbol{\psi}}_i \cup \nu_i^*(\mathbf{y})$) restricts to give $(\overline{\boldsymbol{\psi}}_i + \boldsymbol{\delta}_{ix}) \cup \nu_i^*(\mathbf{y})$ on the twig containing p_i — here x denotes the gluing mark of that twig. Now (on each twig) expand the product of all these restrictions into a sum over s (which accounts for the binomial coefficients). Third, apply formula (3) to write $\boldsymbol{\delta}_{ix} \cup \nu_i^*(\mathbf{y}) = \boldsymbol{\delta}_{ix} \cup \nu_x^*(\mathbf{y})$, with the effect of accumulating evaluation classes on the gluing mark. Finally, by equation (2) we can push down each term involving a diagonal class $\boldsymbol{\delta}_{ix}$ along the forgetful morphism forgetting p_i; the effect is simply deleting $\boldsymbol{\delta}_{ix}$. □

1.6 The tangency quantum potential Γ shall now be defined as the generating function for the first enumerative descendants. Let $\mathbf{x} = \sum x_i T_i$ and $\mathbf{y} = \sum y_i T_i$ be generic even elements of $H^*(X, \mathbb{Q})$, and set

$$\begin{aligned} \Gamma(\mathbf{x}, \mathbf{y}) \quad &:= \sum_{\beta > 0} q^\beta \langle \exp(\overline{\tau}_0(\mathbf{x}) + \overline{\tau}_1(\mathbf{y})) \rangle_\beta \\ &= \sum_{\beta > 0} q^\beta \sum_{a,b} \langle \frac{\overline{\tau}_0(\mathbf{x})^a}{a!} \frac{\overline{\tau}_1(\mathbf{y})^b}{b!} \rangle_\beta . \end{aligned} \tag{5}$$

The inner sum is over *all* non-negative integers a and b — this is meaningful since there is no $\beta = 0$ term in the outer sum. (The degree-zero case is considered and included below, in 4.3 and 4.4.) The coefficients q^β are necessary only to ensure formal convergence; they belong to the Novikov ring Λ (as defined in Getzler [6]), which will be our coefficient ring when treating the tangency quantum potential.

For convenience we identify $\mathbf{x}$ and $\mathbf{y}$ with their coordinate vectors $\mathbf{x} = (x_0, \ldots, x_r)$ and $\mathbf{y} = (y_0, \ldots, y_r)$ with respect to the basis $T_0, \ldots, T_r$. The potential thus belongs to the power series ring $\Lambda[[\mathbf{x}, \mathbf{y}]] = \Lambda[[x_0, \ldots, x_r, y_0, \ldots, y_r]]$, and expands to

$$\Gamma(\mathbf{x}, \mathbf{y}) = \sum_{\beta > 0} q^\beta \sum_{\mathbf{a}, \mathbf{b}} \frac{\mathbf{y}^\mathbf{b}}{\mathbf{b}!} \frac{\mathbf{x}^\mathbf{a}}{\mathbf{a}!} \langle \overline{\boldsymbol{\tau}}_0^\mathbf{a} \, \overline{\boldsymbol{\tau}}_1^\mathbf{b} \rangle_\beta . \tag{6}$$

Here the inner sum is over all pairs of vectors $\mathbf{a} = (a_0, \ldots, a_r)$ and $\mathbf{b} = (b_0, \ldots, b_r)$ of non-negative integers, and we employ multi-index notation, e.g. $\mathbf{a}! = a_0! \cdots a_r!$. For the formal variables $\mathbf{x}$ and $\mathbf{y}$, the multi-index notation is reverse, in order to preserve the signs arising from odd variables, e.g. $\mathbf{x}^\mathbf{a} = x_r^{a_r} \cdots x_0^{a_0}$

The variables $\mathbf{x}$ are the usual formal variables from quantum cohomology, so when $\mathbf{y}$ is set to zero, Γ reduces to the usual (quantum part of the) genus-zero Gromov-Witten potential.

The viewpoint of (6) is advantageous for the sake of extracting the invariants, as well as for checking the validity of certain formal operations on the potential. This task is safely left to the reader and henceforth only the more compact notation of (5) is used.

2 Deformation of the Poincaré metric

While the usual quantum potential is based on the Poincaré metric constants $g_{ij} = \int_{\mathbf{0}} T_i \cup T_j$, the tangency quantum potential relates more naturally to a deformation of them, a certain "metric" with values in $\mathbb{Q}[[\mathbf{y}]] = \mathbb{Q}[[y_0, \ldots, y_r]]$.

2.1 The classical product in the Poincaré metric. The two important structures on $H = H^*(X, \mathbb{Q})$ are the intersection product $\cup$, and the trace map $\int_{\mathbf{0}} : H \to \mathbb{Q}$ which is just integration over the fundamental class of X. Set $g_{ij} = \int_{\mathbf{0}} T_i \cup T_j$ and $g_{ijk} = \int_{\mathbf{0}} T_i \cup T_j \cup T_k$. Let (g^{ij}) be the inverse matrix to (g_{ij}). It is used to raise indices as needed; in particular, with $g_{ij}^k = \sum_e g_{ije}\, g^{ek}$, we have the multiplication formula

$$T_i \cup T_j = \sum_k g_{ij}^k\, T_k. \tag{7}$$

2.2 Intersection polynomials of X. For a generic even element $\mathbf{y} \in H$, (identified with its coordinates with respect to $T_0, \ldots, T_r$ as in 1.6), let $\phi(\mathbf{y}) \in \mathbb{Q}[[\mathbf{y}]]$ be the generating function for the integrals on X,

$$\phi(\mathbf{y}) \;:=\; \int_{\mathbf{0}} \exp(\mathbf{y}) \;=\; \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbf{0}} \mathbf{y}^n.$$

Note that the expansion is $\exp(y_0)$ times a polynomial in $y_1, \ldots, y_r$.

Now set

$$\phi_{ij} := \frac{\partial^2}{\partial y_i \partial y_j}\, \phi = \int_{\mathbf{0}} \exp(\mathbf{y}) \cup T_i \cup T_j$$

and use the matrix (g^{ef}) to raise indices, putting

$$\phi^i{}_j := \sum_e g^{ie}\, \phi_{ej}, \qquad \phi_j{}^i := \sum_e \phi_{je}\, g^{ei}, \qquad \text{and} \qquad \phi^{ij} = \sum_{e,f} g^{ie}\, \phi_{ef}\, g^{fj}. \tag{8}$$

The two first entities appear in the following multiplication formula,

$$\begin{aligned} \exp(\mathbf{y}) \cup T_p \;&=\; \sum_{e,f} T_e\, g^{ef} \int_{\mathbf{0}} T_f \cup \exp(\mathbf{y}) \cup T_p \\ &=\; \sum_e T_e\, \phi^e{}_p(\mathbf{y}), \end{aligned} \tag{9}$$

and similarly $T_q \cup \exp(\mathbf{y}) = \sum_f \phi_q{}^f(\mathbf{y})\, T_f$. (These are identities in $H[[\mathbf{y}]]$.)

2.3 Lemma. Sum formula. *Let $\mathbf{y}'$ and $\mathbf{y}''$ be generic even elements of H. Then*

$$\phi(\mathbf{y}' + \mathbf{y}'') = \sum_{e,f} \phi_e(\mathbf{y}')\, g^{ef}\, \phi_f(\mathbf{y}'').$$

Proof. Write down the exponential series for $X \times X = X' \times X''$,

$$\exp\left(\sum y_i' T_i' + \sum y_i'' T_i''\right)$$

and integrate over the diagonal Δ. Computing this integral using the isomorphism $X \simeq \Delta$ yields the left hand side. On the other hand, computing the integral using the Künneth formula for Δ and the projection formula yields the right hand side. $\square$

This sum formula (and its derivatives) ensures that the expected index-raising rules hold. For example, the useful formula

$$\phi^i{}_k{}^j = \sum_\ell g^i_{k\ell}\, \phi^{\ell j} \tag{10}$$

amounts to $\phi^i{}_k{}^j(\mathbf{0}+\mathbf{y}) = \sum\limits_{\ell,m} \phi^i{}_{k\ell}(\mathbf{0})\, g^{\ell m}\, \phi_m{}^j(\mathbf{y})$.

2.4 The deformed metric. Instead of using the integral $\int_{\mathbf{0}} : H \to \mathbb{Q}$, the new metric is based on the linear map $\int_{\mathbf{y}} : H \to \mathbb{Q}[[\mathbf{y}]]$ defined as

$$\int_{\mathbf{y}} \mathbf{z} \quad := \quad \int_{\mathbf{0}} \exp(-2\mathbf{y}) \cup \mathbf{z}.$$

It is thought of as a deformation of $\int_{\mathbf{0}}$ since we recover this map upon setting $\mathbf{y} = \mathbf{0}$. Now define the new metric (γ_{ij}) by

$$\gamma_{ij} := \gamma_{ij}(\mathbf{y}) := \int_{\mathbf{y}} T_i \cup T_j = \phi_{ij}(-2\mathbf{y}),$$

and adopt the obvious notation $\gamma_{ijk} = \int_{\mathbf{y}} T_i \cup T_j \cup T_k = \phi_{ijk}(-2\mathbf{y})$. Let (γ^{ij}) denote the inverse matrix to (γ_{ij}). Then it follows readily from the sum formula above that

$$\gamma^{ij} = \phi^{ij}(2\mathbf{y}) = \sum_{e,f} \phi^i{}_e\, g^{ef}\, \phi_f{}^j. \tag{11}$$

The sum formula also yields $\sum_e \gamma_{ije}\, \gamma^{ef} = \sum_e \phi_{ije}(-2\mathbf{y})\ \phi^{ef}(2\mathbf{y}) = \phi_{ij}{}^f(\mathbf{0}) = g^f_{ij}$, so the intersection product can be written

$$T_i \cup T_j = \sum_{e,f} \gamma_{ije}\, \gamma^{ef}\, T_f.$$

2.5 Example. For $\mathbb{P}^2$ (with $h := c_1(\mathcal{O}(1))$ and basis $T_i := h^i$), we get

$$(\gamma^{ij}) = \exp(2y_0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2y_1 \\ 1 & 2y_1 & 2y_1^2 + 2y_2 \end{pmatrix}.$$

This matrix (with y_0 and y_2 set to zero) was first written down in the pioneer article [3] of Di Francesco and Itzykson.

3 WDVV equations and the tangency quantum product

3.1 Notation. Let lower indices denote partial derivatives, e.g.

$$\Gamma_{ij} := \Gamma_{x_i x_j} := \frac{\partial^2}{\partial x_i \partial x_j}\Gamma = \sum_{\beta>0} q^\beta \,\langle\, \exp(\overline{\tau}_0(\mathbf{x}) + \overline{\tau}_1(\mathbf{y})) \cdot \overline{\tau}_0(T_i)\, \overline{\tau}_0(T_j)\,\rangle_\beta,$$

and set

$$\Gamma_{(ij)} = \Gamma_{(x_i x_j)} := \sum_{k=0}^{r} g_{ij}^{k}\ \Gamma_{x_k}, \tag{12}$$

— the "directional derivative with respect to the product $T_i \cup T_j = \sum g_{ij}^k T_k$".

3.2 Theorem. *The following form of the WDVV equations holds for the tangency quantum potential.*

$$\Gamma_{(ij)k\ell} + \Gamma_{ij(k\ell)} + \sum_{e,f} \Gamma_{ije}\ \gamma^{ef}\ \Gamma_{fk\ell} \quad = \quad \pm\big(\Gamma_{(jk)i\ell} + \Gamma_{jk(i\ell)} + \sum_{e,f} \Gamma_{jke}\ \gamma^{ef}\ \Gamma_{fi\ell}\big),$$

where $\pm$ denotes the sign $(-1)^{\deg T_i(\deg T_j + \deg T_k)}$.

3.3 Remark. In the special case $X = \mathbb{P}^2$, it turns out the only non-trivial relation is the one with $i = j = 1$ and $k = \ell = 2$. Since $\Gamma_0 = 0$ by the string equation (cf. [7]), the relation then reads

$$\Gamma_{222} = \exp(2y_0)\big(\Gamma_{112}^2 - \Gamma_{111}\Gamma_{122} + 2y_1(\Gamma_{122}\Gamma_{112} - \Gamma_{111}\Gamma_{222}) + (2y_1^2 + 2y_2)(\Gamma_{122}^2 - \Gamma_{112}\Gamma_{222})\big).$$

This equation was first found by L. Ernström and G. Kennedy [4], cf. Remark 3.7 below, while the special case of $y_0 = y_2 = 0$ goes back to Di Francesco-Itzykson [3]. Setting also $y_1 = 0$ we are back to the celebrated formula of M. Kontsevich [10].

Proof. The proof follows the line of arguments of the proof of the WDVV equations for the usual Gromov-Witten potential, cf. Kontsevich-Manin [10]. The novelty is the splitting lemma for enumerative descendants and the appearance of the deformed metric.

For fixed $\beta > 0$ and $a, b \geq 0$, consider the moduli stack $\overline{M}_{0,\{p_1,p_2,p_3,p_4\}\cup S}(X, \beta)$ where S is a marking set of cardinality $a + b$. Consider the product

$$\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\ \frac{\overline{\tau}_0(\mathbf{x})^a}{a!}\frac{\overline{\tau}_1(\mathbf{y})^b}{b!}$$

where the first four classes correspond to the marks p_1, p_2, p_3, p_4. Now integrate the product over each side of the fundamental equivalence

$$(p_1, p_2 \mid p_3, p_4) = (p_2, p_3 \mid p_1, p_4), \tag{13}$$

where $(p_1, p_2 \mid p_3, p_4)$ denotes the sum of all boundary divisors having p_1 and p_2 on one twig and p_3 and p_4 on the other. Summing up these equations over all $a, b \geq 0$ and over

all $\beta > 0$ (as in the definition of Γ) we'll get the desired equation. Let us treat the left hand side. On the right hand side of the equation the arguments are the same; only it is necessary initially to permute the four special factors, which accounts for the sign.

On the left hand side of the equation, let us first consider the contribution from the trivial degree partitions, say $\beta' = 0$. Then the only possible distribution of the marks giving contribution is when all the spare marks fall on the right hand twig, which leaves us with the single boundary divisor D_{12}. Now according to 1.3, the effect of multiplication with this divisor is to merge the two classes $\overline{\tau}_0(T_i)$ and $\overline{\tau}_0(T_j)$, so in the end we get

$$\langle\, \overline{\tau}_0(T_i \cup T_j)\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\frac{\overline{\tau}_0(\mathbf{x})^a}{a!}\frac{\overline{\tau}_1(\mathbf{y})^b}{b!}\,\rangle_\beta.$$

Similarly, the case $\beta'' = 0$ gives $\langle\, \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k \cup T_\ell)\frac{\overline{\tau}_0(\mathbf{x})^a}{a!}\frac{\overline{\tau}_1(\mathbf{y})^b}{b!}\,\rangle_\beta$. Summing over $\beta > 0$ and $a, b \geq 0$ gives exactly the two linear terms on the left hand side of the promised equation, cf. (12).

Now for those boundary divisors in the linear equivalence corresponding to strictly positive degree partitions. To each irreducible component on the left hand side of (13), we apply the Splitting Lemma 1.5, getting all together

$$\sum\sum_{p,q}\langle\, \frac{\overline{\tau}_0(\mathbf{x})^{a'}}{a'!}\frac{\overline{\tau}_1(\mathbf{y})^{b'}}{b'!}\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\,\overline{\tau}_0(\frac{\mathbf{y}^{c'}}{c'!}\cup T_p)\,\rangle_{\beta'}$$

$$g^{pq}\;\langle\, \overline{\tau}_0(T_q \cup \frac{\mathbf{y}^{c''}}{c''!})\,\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\frac{\overline{\tau}_0(\mathbf{x})^{a''}}{a''!}\frac{\overline{\tau}_1(\mathbf{y})^{b''}}{b''!}\,\rangle_{\beta''}.$$

where the big outer sum is over all $a' + a'' = a$ and all $b' + c' + b'' + c'' = b$. (The b' corresponds to what was called $b' - s'$ in the splitting lemma.) Now sum over all a and b getting

$$\sum_{p,q}\langle\, \exp(\overline{\tau}_0(\mathbf{x}) + \overline{\tau}_1(\mathbf{y}))\cdot\overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\,\overline{\tau}_0(\exp(\mathbf{y})\cup T_p)\,\rangle_{\beta'}$$

$$g^{pq}\;\langle\, \overline{\tau}_0(T_q \cup \exp(\mathbf{y}))\,\overline{\tau}_0(T_k)\overline{\tau}_0(T_\ell)\cdot\exp(\overline{\tau}_0(\mathbf{x}) + \overline{\tau}_1(\mathbf{y}))\,\rangle_{\beta''}.$$

Next, use equation (9) to get rid of $\exp(\mathbf{y})$, and sum over all $\beta > 0$ as in the definition of Γ, arriving at

$$\sum_{\substack{p,q\\ e,f}} \Gamma_{ije}\,\phi^e{}_p\,g^{pq}\,\phi_q{}^f\,\Gamma_{fk\ell}.$$

By equation (11), this is just the quadratic term of the left hand side of the desired equation. $\square$

3.4 Topological recursion relation. The above deformed WDVV equations alone are not sufficient to determine all the first enumerative descendants from the primary Gromov-Witten invariants. But topological recursion is available also for the enumerative

descendants cf. [7]. In the present set-up, that topological recursion relation takes the following pleasant form.

$$\Gamma_{y_i x_j x_k} = \Gamma_{x_i(x_j x_k)} - \Gamma_{(x_i x_j)x_k} - \Gamma_{(x_i x_k)x_j} + \sum_{e,f} \Gamma_{x_i x_e}\ \gamma^{ef}\ \Gamma_{x_f x_j x_k}.$$

The shape of this equation stems from the boundary expression of the modified psi class, $\overline{\boldsymbol{\psi}}_i = (p_i \mid p_j, p_k) - \boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_i$ is the sum of all boundary divisors such that p_i is on a contracting twig. The deformed metric enters in the quadratic terms for the same reason as in the proof above.

3.5 The tangency quantum product. The *tangency quantum product* is the $\Lambda[[\mathbf{x},\mathbf{y}]]$-bilinear product on $H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda[[\mathbf{x},\mathbf{y}]]$ defined by the rule

$$T_i * T_j := T_i \cup T_j + \sum_{e,f} \Gamma_{ije}\ \gamma^{ef}\ T_f.$$

Observe that this product specialises to the usual quantum product upon setting the formal variables $\mathbf{y}$ to zero. Clearly the product is supercommutative.

3.6 Theorem. *The tangency quantum product is associative.*

Proof. This is a straightforward consequence of Theorem 3.2; it amounts to checking the associativity relations on the generators, using the definition of the product. The only subtle point in the verification is the identity

$$\sum_{e,f} \Gamma_{ije}\ \gamma^{ef}\ (T_f \cup T_k) = \sum_{\ell,m} \Gamma_{ij(k\ell)}\ \gamma^{\ell m}\ T_m,$$

which follows from the properties of the structure constants g_{ij}^k listed in (7), (10), and (12). □

3.7 Remark. In the special case $X = \mathbb{P}^2$, this product (or a simple change of coordinates of it) was previously constructed via ad hoc methods by L. Ernström and G. Kennedy [4], who also gave a tour de force proof of its associativity. Their construction relies on the space of stable lifts, and seems to be peculiar to the projective plane.

4 Integrability

4.1 The classical potential for X is the generating function for the triple top products

$$\int_{\mathbf{0}} \frac{\mathbf{x}^3}{3!} \quad = \quad \sum_{i,j,k} \frac{x_k x_j x_i}{6} \int_{\mathbf{0}} T_i \cup T_j \cup T_k.$$

By construction, its third derivatives are just g_{ijk}, the structure constants for the cup multiplication (in the Poincaré metric). In usual quantum cohomology, this potential is reinterpreted as the $\beta = 0$ part of the Gromov-Witten potential: a quantum correction (the $\beta > 0$ part) is added to the classical potential, in such a way that the third derivatives of this sum are the structure constants of a new associative product, the quantum product.

4.2 "The tangency classical potential". The tangency quantum product relates to $\int_{\mathbf{y}}$ exactly as the usual quantum product relates to $\int_{\mathbf{0}}$. So let us introduce a potential

$$\Phi^0(\mathbf{x},\mathbf{y}) := \int_{\mathbf{y}} \frac{\mathbf{x}^3}{3!} = \sum_{i,j,k} \frac{x_k x_j x_i}{6} \int_{\mathbf{y}} T_i \cup T_j \cup T_k,$$

which to fit into the picture could be called the tangency classical potential, although it is neither classical, nor has anything particular to do with tangency. By construction, its third derivatives are γ_{ijk}, the structure constants of the cup multiplication, but this time in the deformed metric:

$$T_i \cup T_j = \sum_{e,f} \Phi^0_{ije}\, \gamma^{ef}\, T_f.$$

4.3 The tangency potential, including $\beta = 0$. Introducing the potential

$$\Phi(\mathbf{x},\mathbf{y}) := \Phi^0(\mathbf{x},\mathbf{y}) + \Gamma(\mathbf{x},\mathbf{y}),$$

whose third derivatives are $\Phi_{ijk} = \gamma_{ijk} + \Gamma_{ijk}$, the tangency quantum product can be written

$$T_i * T_j = \sum_{e,f} \Phi_{ije}\, \gamma^{ef}\, T_f,$$

and the WDVV equation 3.2 then takes the usual form

$$\sum_{e,f} \Phi_{ije}\, \gamma^{ef}\, \Phi_{fk\ell} = \pm \sum_{e,f} \Phi_{jke}\, \gamma^{ef}\, \Phi_{fi\ell}.$$

4.4 Interpretation of the degree-zero term. Writing down this potential $\Phi = \Phi^0 + \Gamma$ calls for an interpretation of Φ^0 in terms of some top products on the degree-zero spaces. Unfortunately there is no way to define the modified psi class on $\overline{M}_{0,n}$, if we want it to satisfy the two rules: (i) it should be compatible with pull-back along forgetful morphisms, and (ii) it should satisfy the push-down formula $\pi_* \overline{\boldsymbol{\psi}} = -2$, independent of the number of marks, to give the dilaton equation for modified psi classes (cf. [7]). Basically this is impossible because the 1-pointed spaces do not exist in degree zero. The best one can do is to define the class on a fixed 4-pointed space and then pull it back to the hierarchy lying over this space, but this definition depends on the choice of the three extra marks.

So take a moduli space $\overline{M}_{0,S\cup\{q_1,q_2,q_3\}}(X,0)$ with three distinguished marks. For each of the other marks $p_s \in S$, define the modified psi class $\overline{\boldsymbol{\psi}}_s$ as the pull-back from $\overline{M}_{0,\{p_s,q_1,q_2,q_3\}} \simeq \mathbb{P}^1$ of the class of degree -2. One easily checks that this is equivalent to defining $\overline{\boldsymbol{\psi}}_s := \boldsymbol{\psi}_s - \boldsymbol{\xi}_s$, where $\boldsymbol{\xi}_s$ is the sum of all boundary divisors such that p_s is on a (contracting) twig together with at most one of the distinguished marks. (This description is then compatible with the boundary description of the modified psi class in the $\beta > 0$ case (cf. [7]), since in that case there are no distinguished marks.)

In this setting, define the invariant

$$\langle \frac{\overline{\tau}_0(\mathbf{x})^a}{a!} \frac{\overline{\tau}_1(\mathbf{y})^b}{b!} \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k) \rangle_0$$

in the obvious way, with the last three classes corresponding to the three distinguished marks. Identifying $\overline{M}_{0,S\cup\{q_1,q_2,q_3\}}(X,0)$ with $\overline{M}_{0,S\cup\{q_1,q_2,q_3\}} \times X$, all the evaluation morphisms are just the projection p to X, so the integrand has a factor $p^*(\frac{\mathbf{x}^a}{a!} \cup \frac{\mathbf{y}^b}{b!} \cup T_i \cup T_j \cup T_k)$. The remaining factors are modified psi classes from $\overline{M}_{0,S\cup\{q_1,q_2,q_3\}}$; since each of them is alone on its mark, we can push them down one by one, arriving at a factor $(-2)^b$. The remaining integral $\int 1$ is zero for dimension reasons, unless we have come down to just $\overline{M}_{0,\{q_1,q_2,q_3\}}$, which means $a = 0$. We conclude

$$\langle \frac{\overline{\tau}_1(\mathbf{x})^b}{b!} \overline{\tau}_0(T_i)\overline{\tau}_0(T_j)\overline{\tau}_0(T_k) \rangle_0 = \int_{\mathbf{0}} \frac{(-2\mathbf{y})^b}{b!} \cup T_i \cup T_j \cup T_k.$$

Summing over all b (and a) we get

$$\int_{\mathbf{0}} \exp(-2\mathbf{y}) \cup T_i \cup T_j \cup T_k = \int_{\mathbf{y}} T_i \cup T_j \cup T_k = \gamma_{ijk}$$

showing that at least the third derivatives of Φ^0 have an interpretation as top products on degree-0 moduli spaces, and after all it is the third derivatives that really matter.

Once we know that the structure constants of the tangency quantum product are third derivatives of the single potential Φ, we are in position to given an interpretation in terms of Frobenius manifolds.

4.5 Formal Frobenius manifolds. For convenience let us recall (from Manin [12], Ch. III) the definition of a formal Frobenius manifold (over k). Let k be a supercommutative $\mathbb{Q}$-algebra. Let H be a free k-module of finite rank, with generators $T_0, \dots, T_r$, and let $g : H\otimes H \to k$ denote an even symmetric non-degenerate bilinear pairing. Let $K = k[[H^t]]$ be the completed symmetric algebra of the dual module H^t. In other words, if $\mathbf{x} = \sum x_i T_i$ is a generic even element of H then $K = k[[x_0, \dots, x_r]]$. Now the structure of a formal Frobenius manifold on (H, g) is given by an even potential $\Phi \in K$ (defined up to quadratic terms) satisfying WDVV. In other words, the multiplication $T_i * T_j := \sum_{e,f} \Phi_{ije}\, g^{ef}\, T_f$ makes $H \otimes_k K$ into an associative supercommutative K-algebra.

With $H := H^*(X, \Lambda)$, the results of this section readily imply:

4.6 Theorem. *The cohomology $\Lambda[[\mathbf{y}]]$-module $H[[\mathbf{y}]]$ with bilinear non-degenerate pairing $\gamma : H[[\mathbf{y}]] \otimes H[[\mathbf{y}]] \to \Lambda[[\mathbf{y}]]$, equipped with the tangency quantum potential $\Phi \in \Lambda[[\mathbf{x},\mathbf{y}]]$ constitutes a formal Frobenius manifold over $\Lambda[[\mathbf{y}]]$.* □

In fact, this formal Frobenius manifold is a deformation over $\mathbb{Q}[[\mathbf{y}]]$ of the formal Frobenius manifold of usual quantum cohomology. While the underlying space is trivially deformed under this deformation, the metric and the potential vary non-trivially.

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